

## THRICE-PUNCTURED SPHERES IN HYPERBOLIC 3-MANIFOLDS

BY

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**ABSTRACT.** The work of W. Thurston has stimulated much interest in the volumes of hyperbolic 3-manifolds. In this paper, it is demonstrated that a 3-manifold  $M'$  obtained by cutting open an oriented finite volume hyperbolic 3-manifold  $M$  along an incompressible thrice-punctured sphere  $S$  and then reidentifying the two copies of  $S$  by any orientation-preserving homeomorphism of  $S$  will also be a hyperbolic 3-manifold with the same hyperbolic volume as  $M$ . It follows that an oriented finite volume hyperbolic 3-manifold containing an incompressible thrice-punctured sphere shares its volume with a nonhomeomorphic hyperbolic 3-manifold. In addition, it is shown that two orientable finite volume hyperbolic 3-manifolds  $M_1$  and  $M_2$  containing incompressible thrice-punctured spheres  $S_1$  and  $S_2$ , respectively, can be cut open along  $S_1$  and  $S_2$  and then glued together along copies of  $S_1$  and  $S_2$  to yield a 3-manifold which is hyperbolic with volume equal to the sum of the volumes of  $M_1$  and  $M_2$ . Applications to link complements in  $S^3$  are included.

**1. Introduction.** The work of W. Thurston has shown that many 3-manifolds possess complete hyperbolic structures of finite volume. Although the hyperbolic volume provides a useful invariant for the study of these manifolds, Wielenberg [6] demonstrated that for any positive integer  $N$ , there are  $N$  nonhomeomorphic hyperbolic 3-manifolds all with the same volume. His method was to cut particular finite volume hyperbolic 3-manifolds with known fundamental polyhedra open along totally geodesic thrice-punctured spheres and then reglue the two copies of each thrice-punctured sphere by a particular isometry yielding nonhomeomorphic hyperbolic 3-manifolds with the same volume.

In what follows, it is shown that this phenomenon will always hold true. That is, let  $S$  be an incompressible thrice-punctured sphere in an orientable finite volume hyperbolic 3-manifold  $M$ . Let  $M'$  be the 3-manifold obtained by cutting  $M$  open along  $S$  and then reidentifying the two copies of  $S$  by an orientation-preserving homeomorphism of  $S$ . Then  $M'$  is hyperbolic with the same volume as  $M$ .

In addition, we show that one can cut two finite volume hyperbolic 3-manifolds  $M_1$  and  $M_2$  open along embedded incompressible thrice-punctured spheres  $S_1$  and  $S_2$  contained in  $M_1$  and  $M_2$ , respectively, and then glue copies of the thrice-punctured

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spheres together to yield a hyperbolic 3-manifold  $M$  with volume equal to the sum of the volumes of  $M_1$  and  $M_2$ .

To prove both of the above, we first prove that incompressible thrice-punctured spheres in finite volume hyperbolic 3-manifolds are isotopic to totally geodesic thrice-punctured spheres. This result was known to A. Marden previous to this paper. The above stated results are then proved. Finally, we discuss applications to link complements. These are of particular interest since the above results furnish us with a means to calculate the hyperbolic volumes of many link complements not previously known. In addition, both of these results were originally conjectured on the basis of explicit calculations of hyperbolic volumes for particular link complements.

This paper is based on work completed in my Ph.D. thesis under James W. Cannon at the University of Wisconsin.

**2. Preliminaries.** A finite volume hyperbolic 3-manifold  $M$  is a 3-manifold without boundary that possesses a complete Riemannian metric with finite volume and constant sectional curvature  $-1$ . This is equivalent to the existence of a covering map from hyperbolic 3-space to  $M$  such that the covering translations act as a discrete group of isometries on hyperbolic 3-space and any fundamental polyhedron for the action of the covering translations on hyperbolic 3-space has finite volume. In this particular paper, all of the finite volume hyperbolic 3-manifolds that we consider are noncompact and are therefore the interiors of compact 3-manifolds with nonempty boundaries consisting of tori.

Hyperbolic 3-space is denoted by  $H^3$ . The corresponding sphere at infinity is denoted  $S_\infty^2$ . We denote the orientation-preserving isometries of  $H^3$  by  $\text{Isom}^+(H^3)$ . If  $M$  is a finite volume hyperbolic 3-manifold, we use  $v(M)$  to denote its hyperbolic volume.

A surface embedded in a finite volume hyperbolic 3-manifold  $M$  is totally geodesic if it lifts to the disjoint union of geodesic planes in  $H^3$ .

We will assume the following facts, proofs for which can be found in Thurston [5]:

(i) A finite volume hyperbolic 3-manifold decomposes into a compact piece and a finite set of cusps, each of which is topologically the product of a torus with an open interval and each of which is covered by the disjoint union of horoballs in  $H^3$ .

(ii) Hyperbolic volume is a topological invariant for finite volume hyperbolic 3-manifolds.

(iii) Elements of the fundamental group of a finite volume hyperbolic 3-manifold that are conjugate to elements in the cusp subgroups are exactly the parabolic isometries when the fundamental group acts on  $H^3$ .

**3. Straightening thrice-punctured spheres.** In this section we prove the following

**THEOREM 3.1.** *Let  $M$  be a compact orientable 3-manifold such that  $\mathring{M}$  is a finite volume hyperbolic 3-manifold. Let  $S$  be an incompressible thrice-punctured sphere properly embedded in  $M$ . Then  $S$  is isotopic to a thrice-punctured sphere  $S'$  properly embedded in  $M$  such that  $\mathring{S}'$  is totally geodesic in the hyperbolic structures on  $\mathring{M}$ .*

PROOF. Choose a basepoint  $x_0$  on  $S$ . Since  $\mathring{M}$  is a hyperbolic 3-manifold, we can choose a covering map  $p: H^3 \rightarrow \mathring{M}$  and basepoint  $\tilde{x}_0 \in p^{-1}(x_0)$  which induce a monomorphism  $\phi: \pi_1(M, x_0) \rightarrow \text{Isom}^+(H^3)$ . We will denote the covering translation  $\phi([\alpha])$  by  $T_\alpha$ .

Since each cusp of  $\mathring{M}$  inherits a product structure from  $H^3$ , we can isotope  $S$  so that  $\mathring{S}$  is totally geodesic in the cusps.

Choose simple closed curves  $\alpha, \beta$  and  $\gamma$  on  $S$ , each based at  $x_0$ , such that each is homotopic to a different boundary component of  $S$  and such that  $[\alpha] \cdot [\beta] = [\gamma]$  in  $\pi_1(S, x_0)$ . Since  $[\alpha], [\beta]$  and  $[\gamma]$  all lie in subgroups of  $\pi_1(M, x_0)$  corresponding to cusps,  $T_\alpha, T_\beta$  and  $T_\gamma$  are all parabolic isometries. Using the Upper Half-Space model for  $H^3$ , where we choose  $\{\infty\}$  to correspond to the fixed point of  $T_\alpha$ , we can let

$$T_\alpha = \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}, \quad T_\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for some  $w \neq 0, a, b, c, d \in \mathbb{C}$ , where  $a + d = 2$  and  $ad - bc = 1$ . Then

$$T_\gamma = T_\alpha \cdot T_\beta = \begin{bmatrix} a + cw & b + dw \\ c & d \end{bmatrix}$$

is parabolic so  $a + cw + d = \pm 2$ . Thus either  $c = 0$  or  $c = -4/w$ .

If  $c = 0$ , then  $T_\alpha$  and  $T_\beta$  commute and  $i_*(\pi_1(S, x_0))$  is abelian, where  $i: S \rightarrow M$  is the inclusion map. However, since  $S$  is incompressible,  $i_*\pi_1(S, x_0) \cong \mathbb{Z} * \mathbb{Z}$ .

Thus  $c = -4/w$ . The fixed point of  $T_\beta$ , denoted  $x_\beta$ , is then given by  $w(d - a)/8$ . The fixed point of  $T_\gamma$ , denoted  $x_\gamma$ , is given by  $w(d - a)/8 + w/2$ .

Note that  $T_\alpha$  preserves the circle  $C = \{w(d - a)/8 + tw: t \in \mathbb{R}\} \cup \{\infty\}$ .  $T_\beta$  also preserves  $C$  since

$$\begin{aligned} T_\beta(\infty) &= -wa/4 = w(d - a)/8 - w/4 \in C, \\ T_\beta(x_\gamma) &= T_\alpha^{-1}(x_\gamma) = w(d - a)/8 - w/2 \in C, \\ T_\beta(x_\beta) &= x_\beta. \end{aligned}$$

Hence  $\phi(\pi_1(S, x_0))$  preserves the circle  $C$ .

Let  $P'$  be the hyperbolic plane with limit set  $C$ . A fundamental domain for the action of the group  $\phi(\pi_1(S, x_0))$  on  $P'$  is shown in Figure 1.

We need the following

LEMMA 3.2. *If  $[\lambda] \in \pi_1(M, x_0)$ , then  $T_\lambda(P') \cap P' = \emptyset$  or  $P'$ .*

PROOF. Since  $S$  is incompressible,  $p^{-1}(\mathring{S})$  is the disjoint union of not necessarily geodesic planes in  $H^3$ . If  $S$  is not totally geodesic let  $P$  be the nongeodesic plane in  $p^{-1}(\mathring{S})$  containing  $\tilde{x}_0$ . Note that  $\phi(\pi_1(S, x_0))$  must preserve  $P$ .

Since  $\mathring{S}$  is not compact, a fundamental domain for the action of  $\phi(\pi_1(S, x_0))$  on  $P$  must contain limit points on  $S_\infty^2$ . Let  $y$  be such a limit point. Since  $\phi(\pi_1(S, x_0))$  preserves the limit set of  $P$ , denoted  $L(P)$ ,  $T_\alpha^n(y) \in L(P)$  for all integers  $n$ . But  $T_\alpha^n(y)$  approaches  $x_\alpha$  on  $S_\infty^2$  as  $n$  approaches  $\infty$ , hence  $x_\alpha \in L(P)$ . Similarly,  $x_\beta$  and  $x_\gamma \in L(P)$ . Consequently, since the images of  $x_\alpha, x_\beta$  and  $x_\gamma$  under  $\phi(\pi_1(S, x_0))$  are dense in  $C$ ,  $C \subseteq L(P)$ .

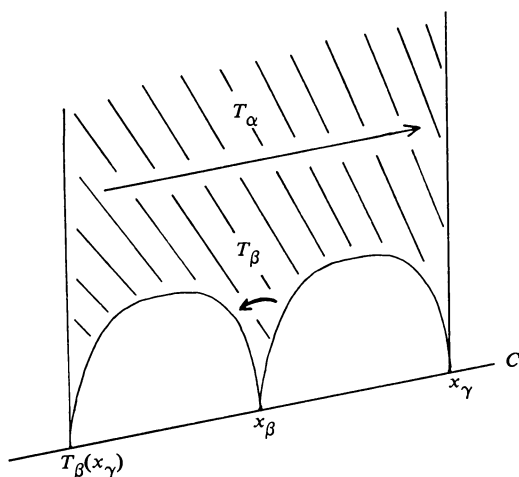


FIGURE 1

In fact,  $C = L(P)$  as if not, let  $x \in L(P)$ ,  $x \notin C$ . Let  $\{x_i\}_{i=1}^\infty$  be a sequence of points on  $P$  converging to  $x$ .

Suppose  $\{p(x_i)\}_{i=1}^\infty$  is contained in a compact submanifold of  $\hat{S}$ . Then  $p(x_i) \rightarrow y \in \hat{S}$ . Lifting a neighborhood of  $y$  in  $\hat{S}$  to  $P$  yields  $g_i(x_i) \rightarrow \tilde{y}$  for some  $g_i \in \phi(\pi_1(S, x_0))$ . Hence  $g_i^{-1}(\tilde{y}) \rightarrow x$ . But the limit points corresponding to the action of  $\phi(\pi_1(S, x_0))$  on any point in  $H^3$  all lie in  $C$ .

Thus  $\{p(x_i)\}_{i=1}^\infty$  is not contained in any compact submanifold of  $\hat{S}$ . Hence there is a subsequence, still denoted  $\{p(x_i)\}_{i=1}^\infty$ , that goes out a particular cusp of  $M$ . There exists a horoball  $B$  about one of  $x_\alpha$ ,  $x_\beta$  and  $x_\gamma$  that projects to this cusp. Without loss of generality, suppose it is  $x_\alpha$ .

Then there is a sequence of elements  $\{g_i\}_{i=1}^\infty$  in  $\phi(\pi_1(M, x_0))$  such that  $\{g_i(x_i)\}_{i=1}^\infty$  is contained in  $B$ , with  $g_i(x_i) \rightarrow x_\alpha$ . Since  $g_i(x_i)$  must lie on a copy of  $P$  and the only copies of  $P$  intersecting  $B$  do so in geodesic planes with limit sets containing  $x_\alpha$ , there exist group elements  $\{\mu_i\}_{i=1}^\infty$  in the cusp subgroup such that  $\mu_i g_i(x_i) \in P$ . Since such an element  $\mu_i$  preserves horospheres about  $x_\alpha$  and since as  $i$  increases,  $g_i(x_i)$  must lie in horospheres of shrinking radii,  $\mu_i g_i(x_i) \rightarrow x_\alpha$ .

However,  $\mu_i g_i \in \phi(\pi_1(S, x_0))$  as it sends a point in  $P$  back to  $P$ . Hence  $x$  is in the limit set of  $\phi(\pi_1(S, x_0))$  acting on the horoball about  $x_\alpha$ . But that limit set is  $C$ , contradicting the fact  $x \notin C$ .

Thus, for any  $[\lambda] \in \pi_1(M, x_0)$ ,  $T_\lambda(P')$  has the same limit set as one of the nongeodesic planes covering  $S$ , namely  $T_\lambda(P)$ . Since any two of the nongeodesic planes are disjoint, their limit sets are either identical or intersect in at most a point, in which case the limit circles are tangent. Hence  $T_\lambda(C) \cap C = \emptyset$ ,  $C$  or one point, and  $T_\lambda(P') \cap P' = \emptyset$  or  $P'$ .  $\square$

Thus,  $p^{-1}(p(P'))$  is the disjoint union of totally geodesic hyperbolic planes. Suppose  $T_\lambda(P') \cap P' = P'$  for some  $T_\lambda$  contained in  $\phi\pi_1(M, x_0)$  but not in  $\phi\pi_1(S, x_0)$ . Then  $T_\lambda(C) = C$ . Since  $C$  is the limit set of  $P$ ,  $x_\alpha$  can only be identified to images of itself under the action of  $\phi\pi_1(S, x_0)$ . Hence  $T_\lambda(x_\alpha) = T_\mu(x_\alpha)$  for some  $T_\mu \in \phi\pi_1(S, x_0)$ . Then  $T_\mu^{-1}T_\lambda(x_\alpha) = x_\alpha$  implying  $T_\mu^{-1}T_\lambda$  is in the cusp subgroup

corresponding to  $x_\alpha$ . Since  $T_\mu^{-1}T_\lambda$  preserves  $C$ ,  $T_\mu^{-1}T_\lambda$  must equal  $T_\alpha^n$  for some integer  $n$ . This implies  $T_\lambda \in \phi\pi_1(S, x_0)$ , contradicting our choice of  $T_\lambda$ .

Thus,  $P'$  projects to the interior of a thrice-punctured sphere  $S'$  in  $M$ . Note that each of the boundary components of  $S$  and  $S'$  which lie on the same boundary component of  $M$  are isotopic simple closed curves since both correspond to the same element of the fundamental group of that boundary component of  $M$ .

We now complete the proof of Theorem 3.1 by showing that  $S$  is isotopic to  $S'$  in  $M$ .

Isotope  $S$  into general position with respect to  $S'$ . We will first isotope  $S$  to be disjoint from  $S'$ . Since  $\partial S$  can be isotoped to be disjoint from  $\partial S'$ , all the intersection curves are simple closed curves.

Choose an innermost trivial intersection curve on  $S$  (or  $S'$ ). If it is also trivial on  $S'$  (or  $S$ ), we can remove the intersection by the irreducibility of  $M$ . If it is nontrivial on  $S'$  (or  $S$ ), it is isotopic on  $S'$  (or  $S$ ) to a boundary component of  $S'$  (or  $S$ ). Hence, there is a simple closed curve in  $\partial M$  which is nontrivial in  $\partial M$  but bounds a disk in  $M$ . This contradicts the injection of the cusp subgroups in a finite volume hyperbolic 3-manifold.

Now all remaining intersection curves are both isotopic on  $S$  to a boundary component of  $S$  and isotopic on  $S'$  to a boundary component of  $S'$ . Let  $\alpha$  be such an intersection curve which is the nearest intersection curve to  $\partial S'$  on  $S'$ .

Let  $A_1$  be the annulus on  $S'$  bounded by  $\alpha$  and the one component of  $\partial S'$ . Let  $A_2$  be the annulus on  $S$  bounded by  $\alpha$  and one component of  $\partial S$ . Let  $A = A_1 \cup A_2$ .

Since a hyperbolic 3-manifold does not contain any incompressible,  $\partial$ -incompressible annuli,  $A$  must  $\partial$ -compress. By irreducibility, this implies there is an annulus  $A'$  in  $\partial M$  such that  $A \cup A'$  bounds a solid torus  $V$  in  $M$ , where the  $\partial A$  curves are  $(p, 1)$  curves on  $\partial V$ . Consequently, we can isotope  $A_2$  to  $A_1$  through  $V$ , eliminating the intersection.

We now have  $S \cap S' = \emptyset$ . Lifting  $\mathring{S}$  and  $\mathring{S}'$  to  $H^3$ , we find  $P' \cap T_\mu(P) = \emptyset$  for all  $T_\mu \in \phi\pi_1(M, x_0)$ . Hence  $P' \cup P$  bounds an open 3-cell  $W$  in  $H^3$  such that

$$W \cap T_\mu(W) = \begin{cases} W & \text{if } T_\mu \in \phi\pi_1(S, x_0), \\ \emptyset & \text{if } T_\mu \in \phi\pi_1(M, x_0) \text{ but } T_\mu \notin \phi\pi_1(S, x_0). \end{cases}$$

Let  $p': H^3 \rightarrow H^3/\phi\pi_1(S, x_0)$  and  $M' = p'(W)$ .  $M'$  is contained in a compact manifold  $M''$  which is homeomorphic to a compact submanifold of  $M'$  obtained by removing the images under  $p'$  of small open horoballs about  $x_\alpha$ ,  $x_\beta$  and  $x_\gamma$ .

Let  $F'$  be the thrice-punctured sphere in  $\partial M''$  such that  $P'$  projects to  $\mathring{F}'$ . Note that  $i_*: \pi_1(F') \rightarrow \pi_1(M'')$  is an isomorphism.

By Hempel [2, Theorem 10.2],  $M''$  is homeomorphic to  $F' \times I$  by a homeomorphism taking  $F'$  to  $F' \times 0$ . Corresponding to the projection of  $P$ , there is a thrice-punctured sphere  $F$  in  $\partial M''$  such that  $\mathring{F}' \cup \mathring{F} = \partial M'$  and  $F' \cup F \cup \bigcup_{i=1}^3 A_i = \partial M''$ , where each  $A_i$  is an annulus in  $\partial M''$  bounded by a component of  $\partial F$  and a component of  $\partial F'$ . Hence  $F$  can be isotoped through  $M''$  to  $F'$ .

This isotopy lifts to yield an isotopy of  $P$  to  $P'$  through  $W$  which is equivariant with respect to  $\phi\pi_1(S, x_0)$ . Since  $W \cap T_\mu W = \emptyset$  for  $T_\mu \notin \phi\pi_1(S, x_0)$ , this isotopy is

actually equivariant with respect to  $\phi\pi_1(M, x_0)$  and therefore projects to an isotopy of  $\mathring{S}$  to  $\mathring{S}'$  through  $\mathring{M}$ . This isotopy extends to the boundaries.  $\square$

**4. Statement and proof of the main results.** Let  $S$  be a properly embedded incompressible thrice-punctured sphere in a compact oriented 3-manifold  $M$ . Let  $M' = M - N(S)$ , where  $N(S)$  is a regular neighborhood of  $S$  in  $M$ . Let  $S_0$  and  $S_1$  be the two copies of  $S$  in  $\partial M'$ .

Let  $\mu: S_0 \rightarrow S_1$  be the identification map that would give us  $M$  back again, and let  $\lambda: S_1 \rightarrow S_1$  be any orientation preserving homeomorphism. Let  $M''$  be the 3-manifold obtained from  $M'$  by identifying  $S_0$  and  $S_1$  by the identification map  $\lambda \circ \mu$ .

**THEOREM 4.1.** *If  $\mathring{M}$  is a finite volume hyperbolic 3-manifold, then so is  $\mathring{M}''$  and  $v(\mathring{M}) = v(\mathring{M}'')$ .*

**PROOF.** By Theorem 3.1, we can assume  $\mathring{S}$  is totally geodesic. Since any orientation-preserving homeomorphism of  $S_1$  is the composite of orientation-preserving homeomorphisms that preserve one boundary component of  $S_1$  and switch the other two, it is enough to prove the theorem for homeomorphisms of this type. Without loss of generality, we will assume  $\lambda$  switches the  $\partial$ -components of  $S_1$  corresponding to  $[\alpha]$  and  $[\beta]$  while preserving the  $\partial$ -component corresponding to  $[\gamma]$ , where  $[\alpha]$ ,  $[\beta]$  and  $[\gamma]$  are defined as in the proof of Theorem 3.1.

Let  $P$  be the hyperbolic plane in  $H^3$  which projects to  $S$  when  $\phi\pi_1(S, x_0)$  acts on it. Let  $D$  be the fundamental domain for the action of  $\phi\pi_1(S, x_0)$  on  $P$  as in Figure 1. Let  $\tilde{x}_0$  be the intersection of the geodesic  $g_1$  running from  $x_\beta$  to  $x_\alpha$  in  $D$  and the geodesic  $g_2$  running from  $x_\gamma$  to  $T_\beta(x_\gamma)$  in  $D$ .

Define a fundamental domain  $\Omega$  for  $\phi\pi_1(M, x_0)$  as follows:

$$\Omega = \left\{ x \in H^3 : d(x, \tilde{x}_0) \leq d(x, T_\mu(\tilde{x}_0)) \text{ for all } T_\mu \in \phi\pi_1(S, x_0) \right. \\ \left. \text{and } d(x, P) \leq d(x, T_\theta(P)) \text{ for all } T_\theta \in \phi\pi_1(M, x_0) \right\}.$$

It is not difficult to check that  $\Omega$  is a fundamental domain for  $\phi\pi_1(M, x_0)$ .

**LEMMA 4.2.** *Let  $C$  be any compact set in  $H^3$ . Then only finitely many images of  $\Omega$  by elements in  $\phi\pi_1(M, x_0)$  intersect  $C$ .*

**PROOF.** Suppose  $\{T_i\}_{i=1}^\infty$  is an infinite set of elements in  $\phi(\pi_1(M, x_0))$  such that  $T_i(\Omega)$  intersects  $C$ . Then, after replacing  $\{T_i\}_{i=1}^\infty$  with a subsequence if necessary, there exists  $x \in C$  such that  $d(x, T_i(\Omega)) < 1/i$ .

By the definition of  $T_i(\Omega)$ , it follows that

- (1)  $d(x, T_i(\tilde{x}_0)) \leq d(x, T_i T_\gamma(\tilde{x}_0)) + 2/i$  for all  $T_\gamma \in \phi(\pi_1(S, x_0))$ ,
- (2)  $d(x, T_i(P)) \leq d(x, T_i T_\mu(P)) + 2/i$  for all  $T_\mu \in \phi(\pi_1(M, x_0))$ .

Let  $D = \{y : d(x, y) \leq d(x, T_1(\tilde{x}_0)) + 2\}$ . Suppose there exists a subsequence  $\{T_j'\}_{j=1}^\infty$  such that all of the  $T_j'$ 's are in the same left coset of  $\phi\pi_1(S, x_0)$  in  $\phi\pi_1(M, x_0)$ . Assuming  $T_1$  is in the coset, (1) implies  $T_j'(\tilde{x}_0)$  is in  $D$  for all  $j$ , contradicting discreteness. Hence, by again taking a subsequence, we can assume all the  $T_i$ 's are in distinct left cosets.

Condition (2) then implies each  $T_i(P)$  intersects  $D$  by taking  $T_\mu = T_i^{-1}T_1$ . Hence by taking another subsequence, there exists  $y \in D$  such that  $d(T_i^{-1}(y), P) < 1/i$ .

For each  $T_i^{-1}(y)$ , choose  $T'_i \in \phi\pi_1(S, x_0)$  such that

$$d(T'_i T_i^{-1}(y), \tilde{x}_0) \leq d(T'_i T_i^{-1}(y), T_\gamma(\tilde{x}_0))$$

for all  $T_\gamma \in \phi\pi_1(S, x_0)$ . We are merely choosing  $T'_i$  so that  $T'_i T_i^{-1}(y)$  lies in that fundamental domain of  $\phi\pi_1(S, x_0)$  defined by this condition. Because all the  $T'_i$ 's are in distinct cosets of  $\phi\pi_1(S, x_0)$ , all the  $T'_i T_i^{-1}$ 's are distinct.

Since  $\phi\pi_1(M, x_0)$  is discrete, the sequence  $\{T'_i T_i^{-1}(y)\}_{i=1}^\infty$  must approach  $S_\infty^2$ . But since the sequence stays in a fundamental domain for  $\phi\pi_1(S, x_0)$  centered about  $\tilde{x}_0$  and approaches the plane  $P$ , a subsequence must approach one of the four points where the fundamental domain intersected with  $P$  touches  $S_\infty^2$  (either  $x_\alpha$ ,  $x_\beta$ ,  $x_\gamma$  or  $T_\beta(x_\gamma)$  in Figure 1).

However, far enough out toward one of these points on  $S_\infty^2$ ,  $\phi\pi_1(M, x_0)$  will only identify points in the same horosphere. Since  $T'_i$  leaves  $P$  invariant and  $d(T_i^{-1}(y), P) < 1/i$  we know  $T'_i T_i^{-1}(y)$  stays within  $1/i$  of  $P$ . Thus  $\{T'_i T_i^{-1}(y)\}$  cannot all lie on the same horosphere for all  $i$  greater than some  $N$ , hence we have a contradiction.  $\square$

LEMMA 4.3.  $\Omega$  is finite-sided.

PROOF. In [3, §4.4], Marden points out that whenever our manifold has a finite volume hyperbolic structure a fundamental polyhedron  $R$  is finite-sided provided:

- (i)  $R$  has the appropriate face-pairing properties.
- (ii) A finite number of images of  $R$  under the group  $G$  intersect any given compact set in  $H^3$ .
- (iii) If  $p \in R$  is a parabolic fixed point, then  $R$  is contained in a finite number of images of  $C_p(O')$  under  $M_p$  for some  $O' \in H^3$ , where  $M_p$  is the maximal parabolic subgroup of  $G$  fixing  $p$  and  $C_p(O') = \{x \in H^3: d(x, O') \leq d(x, T(O')) \text{ for all } T \in M_p\}$ .

Condition (i) means the pairs of faces of  $R$  should be identified by elements of the group and each face should lie in a hyperbolic plane. This is certainly true of  $\Omega$ .  $\Omega$  also satisfies (ii) by Lemma 4.2.

To check (iii), put  $p$  at  $\{\infty\}$  in the Upper Half-Space model. Note  $\Omega \subset C'_p(\tilde{x}_0)$ , where  $C'_p(\tilde{x}_0) = \{x: d(x, \tilde{x}_0) \leq d(x, T(\tilde{x}_0)) \text{ for all } T \in M_p \cap \phi(\pi_1(S, x_0)) \text{ and } d(x, P) \leq d(x, Q(P)) \text{ for all } Q \in M_p\}$ .

But since  $M_p$  is isomorphic to  $\mathbf{Z} \oplus \mathbf{Z}$  for any finite volume hyperbolic 3-manifold,  $C'_p(\tilde{x}_0) = D \times (0, \infty)$  for some polygon  $D \subset \mathbf{R}^2$ .  $C_p(\tilde{x}_0) = D' \times (0, \infty)$  for some polygon  $D' \subset \mathbf{R}^2$ . Hence, finitely many images of  $C_p(\tilde{x}_0)$  under  $M_p$  will cover  $C'_p(\tilde{x}_0)$ . This completes the proof of Lemma 4.3 since  $\Omega \subset C'_p(\tilde{x}_0)$ .  $\square$

Since for all  $T_\mu \in \phi\pi_1(M, x_0)$  but  $T_\mu \notin \phi\pi_1(S, x_0)$  we have  $T_\mu(P) \cap P = \emptyset$ , it follows that  $P \cap \Omega = \{x \in P: d(x, \tilde{x}_0) \leq d(x, T_\theta(\tilde{x}_0)) \text{ for all } T_\theta \in \phi\pi_1(S, x_0)\}$ . This is again the fundamental domain for the action of  $\phi\pi_1(S, x_0)$  on  $P$  pictured in Figure 1.

We can cut  $\Omega$  open along  $P \cap \Omega$ , obtaining two components  $\Omega_1$  and  $\Omega_2$ . Since a thrice-punctured sphere cannot separate an oriented 3-manifold with toral boundary

components, there exists a face on  $\Omega_2$  which is identified to a face on  $\Omega_1$ . Let  $T_\lambda$  be the identifying isometry.

Let  $\Omega' = \Omega_1 \cup T_\lambda(\Omega_2)$ . Let  $F_1$  be the copy of  $P \cap \Omega$  on  $\partial\Omega_1$  and  $F_2 = T_\lambda(F_1)$ .  $\Omega'$  is a new fundamental domain for  $\phi\pi_1(M, x_0)$  containing  $F_1$  and  $F_2$  on its boundary. Note that all the edges on  $\partial F_1$  and  $\partial F_2$  correspond to dihedral angles  $\pi/2$  in  $\Omega'$ .

Let  $g$  be the geodesic intersecting the hyperbolic plane containing  $F_2$  at  $T_\lambda(\tilde{x}_0)$  and perpendicular to that plane. Let  $Q$  be the elliptic isometry corresponding to  $180^\circ$  rotation about  $g$ . Note  $Q(F_2) = F_2$  by our choice of  $\tilde{x}_0$ .

Define a new set of identifications on  $\Omega'$  to be the old identifications on all pairs of faces except  $F_1$  and  $F_2$ , and to identify  $F_1$  with  $F_2$  by  $Q \circ T_\lambda$ . Since any homeomorphism of a thrice-punctured sphere is isotopic to a unique isometry in the hyperbolic structure on that thrice-punctured sphere, performing these identifications on  $\Omega'$  will yield a manifold homeomorphic to  $M''$ .

By Maskit [4], all the conditions for this to be the fundamental polyhedron of a discrete hyperbolic group are satisfied with the possible exception of the edge and completeness conditions.

Since  $\Omega'$  with the original identifications is a fundamental polyhedron for  $\phi(\pi_1(M, x_0))$ , the edge and completeness conditions for edges and cusps that do not involve  $F_1$  and  $F_2$  will still be satisfied under the new identifications. In the original identifications, the eight edges on  $\partial F_1 \cup \partial F_2$  consisted of two classes of four edges each, where all edges in the same class were identified to a single edge in the final manifold. Our change in the identification of  $F_1$  and  $F_2$  just switches two edges in one class for two edges in the other. Since all the dihedral angles for these edges are  $\pi/2$ , the condition that the sum of the angles around an edge in the final manifold must be  $2\pi$  is still satisfied.

We still need to satisfy the cycle condition on edges; that is, if we form a product of identifying isometries which preserves an edge on  $\partial F_1$  or  $\partial F_2$ , we need to know that the product is the identity on the edge.

Up to conjugacy and inverse, the only two such products involving edges from  $\partial F_1$  and  $\partial F_2$  are

$$(QT_\lambda)^{-1}(T_\lambda T_\beta T_\lambda^{-1})(QT_\lambda)T_\alpha \quad \text{and} \quad (QT_\lambda)^{-1}(T_\lambda T_\alpha T_\lambda^{-1})(QT_\lambda)T_\beta.$$

Using the fact  $QT_\lambda T_\alpha T_\lambda^{-1}Q = T_\lambda T_\beta T_\lambda^{-1}$ , both of these products are immediately seen to be the identity. Hence, all we have left to check is completeness for each of the cusps on which  $S$  has a boundary component.

Maskit points out that it is enough to check that each tangency vertex transformation is parabolic. (A tangency vertex transformation is a product of face-identifying isometries that fixes a point on  $S_\infty^2$  in the fundamental polyhedron.) In fact, in our case we need only check that one tangency vertex transformation for each cusp is parabolic, since the cusp subgroups are abelian and therefore any other tangency vertex transformation corresponding to this cusp is conjugate to an element that commutes with this parabolic element and is therefore parabolic itself.

However, if we take that element of the cusp subgroup corresponding to the boundary component of  $S$ , it is unaffected by the change in identifications and

hence remains parabolic. Consequently, our fundamental polyhedron with the new identifications does correspond to a discrete hyperbolic group with the same volume as the original group.  $\square$

**COROLLARY 4.4.** *Let  $M$  be an oriented 3-manifold such that  $\mathring{M}$  is a finite volume hyperbolic 3-manifold. Then if  $M$  contains a properly embedded incompressible thrice-punctured sphere  $S$ , there exists an orientable 3-manifold  $M'$  which is not homeomorphic to  $M$  such that  $\mathring{M}'$  is a hyperbolic 3-manifold with the same volume as  $\mathring{M}$ .*

**PROOF.** Cut  $M$  open along  $S$ . If the boundary components of  $S$  do not all lie on the same boundary component of  $M$ , choose a homeomorphism  $\phi$  of  $S$  which interchanges two boundary components of  $S$  lying on distinct boundary components of  $M$  while fixing the third. Then reidentifying the two copies of  $S$  by  $\phi$  will yield a 3-manifold such that its interior is hyperbolic with the same volume but with one less cusp.

If all three of the boundary components of  $S$  lie on the same boundary component  $Q'$  of  $M$ , then one of the three components  $Q$  of  $Q' - \partial S$  intersects both sides of  $S$ . Choose a homeomorphism  $\phi$  of  $S$  which interchanges the two boundary components of  $S$  which bound  $Q$  while fixing the third. Then reidentifying the two copies of  $S$  by  $\phi$  will again yield a 3-manifold such that its interior is hyperbolic with the same volume but with an additional cusp.  $\square$

Let  $S_1$  and  $S_2$  be incompressible thrice-punctured spheres properly embedded in compact orientable 3-manifolds  $M_1$  and  $M_2$ , respectively. Let  $M'_i = M_i - N(S_i)$ . Let  $S_i^0$  and  $S_i^1$  be the two copies of  $S_i$  in  $\partial M'_i$ .

Let  $\lambda_0: S_1^0 \rightarrow S_2^0$  and  $\lambda_1: S_1^1 \rightarrow S_2^1$  be any two homeomorphisms that either both preserve orientations or both reverse orientations. Let  $M$  be the 3-manifold obtained from  $M_1$  and  $M_2$  by identifying  $S_1^0$  and  $S_2^0$  using  $\lambda_0$  and identifying  $S_1^1$  and  $S_2^1$  using  $\lambda_1$ .

**THEOREM 4.5.** *If  $\mathring{M}_1$  and  $\mathring{M}_2$  are finite volume hyperbolic 3-manifolds, then so is  $\mathring{M}$  and  $v(\mathring{M}) = v(\mathring{M}_1) + v(\mathring{M}_2)$ .*

**PROOF.** Exactly as in the proof of Theorem 4.1, we find fundamental polyhedra  $\Omega_1$  and  $\Omega_2$  for  $M_1$  and  $M_2$  such that there is a pair of faces  $F_i^0, F_i^1$  in  $\partial M'_i$  corresponding to the thrice-punctured sphere  $S_i$ . Let  $Q_i: F_i^0 \rightarrow F_i^1$  be the homeomorphism that would again yield  $M_i$ .

Note that it is enough to prove the theorem for any particular pair of homeomorphisms  $\lambda_0$  and  $\lambda_1$  by Theorem 4.1. There exists an orientation-preserving isometry  $T$  of hyperbolic space sending  $F_1^0$  to  $F_2^0$  such that  $T(\Omega_1) \cap \Omega_2 = F_2^0$ . This is true since we can take three of the four points where  $F_1^0$  intersects  $S_\infty^2$  to any other three points on  $S_\infty^2$  by some orientation-preserving hyperbolic isometry. The fourth point will go to the right place by the symmetry of  $F_1^0$  and  $F_2^0$ .

Let  $\Omega = T(\Omega_1) \cup \Omega_2$ . Conjugating the identifying isometries for  $\Omega_1$  gives us the identifications for faces of  $T(\Omega_1)$  with the exception of  $T(F_1^0)$ . The original identifying isometries for faces of  $\Omega_2$  give us the identifications for faces of  $\Omega$  coming from  $\Omega_2$ , with the exception of  $F_2^0$ . Identify  $T(F_1^1)$  to  $F_2^1$  by  $Q_2 T Q_1^{-1} T^{-1}$ . Exactly as

in the proof of Theorem 4.1, we check that the edge and completeness conditions are satisfied.  $\square$

**5. Applications to link complements.** Many link complements containing thrice-punctured spheres are known to be hyperbolic (see, for instance, [1]). Hyperbolicity of the link complements imply the thrice-punctured spheres are incompressible. Restricting the homeomorphisms used to identify copies of the thrice-punctured spheres, Theorems 4.1 and 4.5 yield the following corollaries.

**COROLLARY 5.1.** *Let  $L$  be a link in  $S^3$  such that  $S^3 - L$  is hyperbolic and  $L$  has a projection for which some part appears as in Figure 2(a). Let  $L'$  be the link obtained by replacing that part of the projection of  $L$  appearing in Figure 2(a) with that appearing in Figure 2(b). Then  $S^3 - L'$  is hyperbolic with the same volume as  $S^3 - L$ .*



FIGURE 2

Note that  $S^3 - L'$  is obtained by cutting  $S^3 - L$  open along the twice-punctured disk bounded by the trivial component shown in Figure 2(a), twisting a half twist and reidentifying. A full twist would have yielded a manifold homeomorphic to  $S^3 - L$ .

**COROLLARY 5.2.** *Let  $L_1$  and  $L_2$  be links in  $S^3$  such that  $S^3 - L_1$  and  $S^3 - L_2$  are hyperbolic and  $L_1$  and  $L_2$  have projections as in Figure 3(a). Let  $L$  be the link with projection as in Figure 3(b). Then  $S^3 - L$  is hyperbolic and  $v(S^3 - L) = v(S^3 - L_1) + v(S^3 - L_2)$ .*

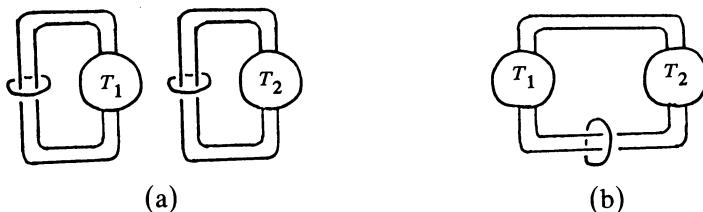


FIGURE 3



Volume = 3.6638...

FIGURE 4

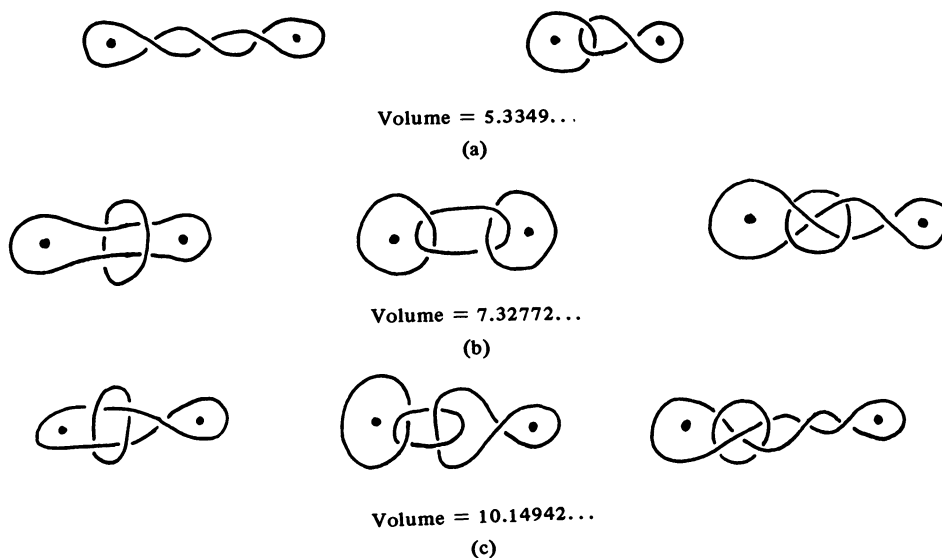


FIGURE 5

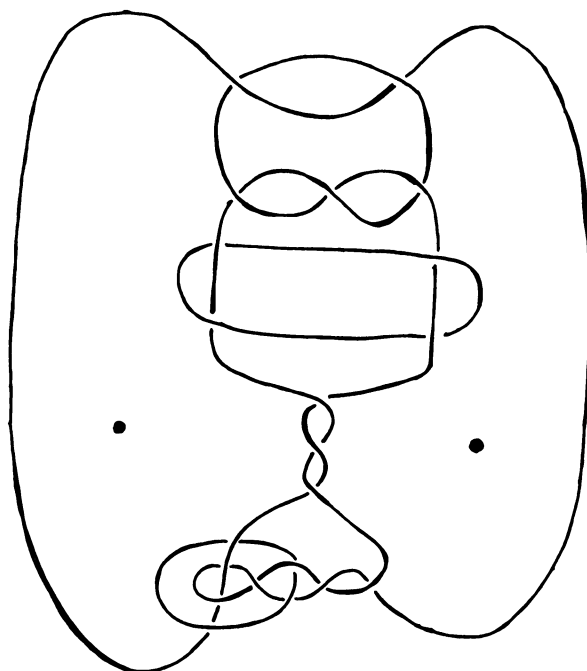


FIGURE 6

We call a link  $L$  formed out of two links  $L_1$  and  $L_2$  as in Corollary 5.2, the *belted sum* of  $L_1$  and  $L_2$ . In each of the Figures 4, 5 and 6, two dots in the projection plane denote a trivial component of the link intersecting the projection plane in only these dots. This trivial component bounds a thrice-punctured disk as in Figure 2(a). For example, Figure 4 denotes the Whitehead link.

Note that the Borromean rings are the belted sum of two copies of the Whitehead link and hence have volume equal to  $7.32772\dots$ . Utilizing Corollary 5.1 and calculations of volumes from Thurston [5], some volumes of link complements are given in Figure 5.

Thus for example, the link appearing in Figure 6, working from top to bottom, is the belted sum of a Whitehead link with a link as in Figure 5(a), a link as in Figure 5(b), a twisted strand which contributes no volume by Corollary 5.1 and a link which is itself the belted sum of a link as in Figure 5(a) and a link as in Figure 5(c). Hence the volume of the complement of this link is given by

$$3.6638\dots + 5.3349\dots + 7.3277\dots + 0 + (5.3349\dots + 10.1494\dots) = 31.8107\dots$$

Similarly, many of the volumes of link complements in the tables can be easily computed.

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